QUASI-STEADY EVAPORATION OF A DROP WITH INTERNAL HEAT RELEASE

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We consider the problem of evaporation or growth of a drop with uniformly distributed internal heat sources, taking into account heat exchange with the surrounding medium. We assume that the Reynolds number $R = ua/\nu$ and the Peclet numbers $P_D = ua/D$, $P_\chi = ua/\chi$ (a is the radius of the drop, u is the velocity of relative motion, ν, D, and χ are the kinematic viscosity, diffusion coefficient, and thermal diffusivity of the vapor-gas medium) are small enough for the distributions of vapor content and temperature to be spherically symmetrical. The Maxwell-Langmuir formula for the rate of diffusion evaporation [1] is extended to the case where internal heat sources are present and energy transfer occurs in the radiation-absorbing vapor-gas medium. The radiation mean free path significantly exceeds the drop radius. We determine the variation of the radius with time.

1. Diffusion Equation and Evaporation Rate. The surface of a drop with $r = a(t)$ divides the whole space into two regions — an inner one and an outer one. All the quantities relating to the inner region are denoted by symbols with a dash, and all those relating to the outer region have no dash. Quantities relating to the interface have the subscript a, those relating to the liquid or vapor have the subscript 1, and those to the gas have the subscript 2. Overall quantities have no subscript. For instance, the total number of molecules in unit volume is $n = n_1 + n_2$. Let m be the mass of the molecule, ρ the density, and ν the radial component of the velocity of the medium. Then

$$\rho_1 = m_1 n_1, \qquad \rho_2 = m_2 n_2, \qquad \rho = \rho_1 + \rho_2, \qquad \rho v = \rho_1 v_1 + \rho_2 v_2 . \tag{1.1}$$

The continuity equations have the form

$$\left(\frac{d}{dr} + \frac{2}{r}\right)\rho_1 v_1 = 0, \qquad \left(\frac{d}{dr} + \frac{2}{r}\right)\rho_2 v_2 = 0 . \tag{1.2}$$

For the inner region $n_1' \gg n_2'$ and, hence, $\rho' = \rho_1'$. In addition, assuming that $\rho'\nu' \gg \rho\nu$ we can ignore the motion of the liquid inside the drop.

According to Chapman and Enskog's theory [2], we can write the diffusion equation for a binary mixture at constant pressure in the absence of external forces and with thermal diffusion ignored:

$$v_1 - v_2 = -\frac{n^2 D_0}{n_1 n_2} \frac{d}{dr} \frac{n_1}{n} . \tag{1.3}$$

Using the boundary condition for $r = a(t)$, $\nu_2 = \dot{a}$ ($\dot{a}$ is the velocity of the phase interface) we obtain the following expression for the vapor flux density on the drop surface:

$$I = \rho_{1a}(v_{1a} - \dot{a}) = -\left(\frac{\rho_1 n^2 D_0}{n_1 n_2} \frac{d}{dr} \frac{n_1}{n}\right)_a \qquad \left(\dot{a} = \frac{da}{dt}\right). \tag{1.4}$$

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Thus, to calculate the velocity of the phase interface, equal to

$$a^{\cdot} = \frac{1}{\rho'}\left(\frac{\rho_1 n^2 D_0}{n_1 n_2}\frac{d}{dr}\frac{n_1}{n}\right)_a, \tag{1.5}$$

we need to solve the diffusion equation.

It follows from the continuity Eqs. (1.2) that

$$\rho_1 v_1 + \rho_2 v_2 = (\rho_{1a} v_{1a} + \rho_{2a}\dot{a})\, a^2 / r^2 . \tag{1.6}$$

Then, eliminating ν_2 from (1.3) and (1.6) we obtain

$$\rho v_1 = (\rho_{1a} v_{1a} + \rho_{2a}\dot{a})\frac{a^2}{r^2} - \frac{\rho_2 n^2 D_0}{n_1 n_2}\frac{d}{dr}\frac{n_1}{n} . \tag{1.7}$$

If we substitute (1.7) in the first of Eqs. (1.2) and neglect the terms containing the small parameter ρ/ρ', the diffusion equation takes the form

$$\left(\frac{d}{dr} + \frac{2}{r}\right)\left[\frac{\rho_1\rho_2 n^2 D_0}{\rho n_1 n_2}\frac{d}{dr}\frac{n_1}{n} + \frac{\rho^2 a^2}{\rho r^2}\left(\frac{\rho_1 n^2 D_0}{n_1 n_2}\frac{d}{dr}\frac{n_1}{n}\right)_a\right] = 0 . \tag{1.8}$$

Let the vapor density at infinity be ρ_∞ and on the drop surface be equal to the saturated vapor density ρ_a at surface temperature T_a. Assuming that $|\rho_a - \rho_\infty| \ll \rho_a$ we can confine ourselves in Eq. (1.8) to terms of the first order of smallness in the parameter $(\rho_a - \rho_\infty)/\rho_\infty$. In addition, since Eq. (1.8) was obtained by neglect of thermal and pressure diffusion effects, n in this equation can be assumed to be independent of r;

$$\left(\frac{d}{dr} + \frac{2}{r}\right)\frac{d\rho_1}{dr} = 0 . \tag{1.9}$$

The solution of this equation enables us to calculate the vapor flux density on the drop surface,

$$j = -\frac{\rho D}{\rho_2}\left(\frac{d\rho_1}{dr}\right)_a \qquad \left(D = \frac{m_2 n}{\rho} D_0\right). \tag{1.10}$$

The saturated vapor density ρ_a is a known function of temperature. In the case of a small temperature difference this function can be approximated by a linear relationship

$$\rho_a = \rho_{s\infty}[1 + \beta(T_a - T_\infty)/T_\infty] . \tag{1.11}$$

The temperature difference is determined from the solution of the heat problem.

2. Energy Equation. We assume that energy transfer from the drop to the gas is due to radiation, diffusion, and heat conduction.

Since, according to the conditions of the problem, the radiation mean free path significantly exceeds the drop radius we can assume that radiation has no effect on the temperature distribution in the vicinity of the drop.

It is obvious that the flux of radiant energy in this vicinity is

$$S_r = \varepsilon\sigma(T_a^4 - T_\infty^4)\, a^2/r^2 \simeq 4\varepsilon\sigma T_\infty^3 (T_a - T_\infty)\, a^2/r^2 , \tag{2.1}$$

where σ is the Stefan-Boltzmann constant and ε is the effective emissivity of a drop surrounded by vapor.

The energy flux transferred by diffusion due to the difference in enthalpies of the diffusing substances is, as is known [2],

$$S_D = {}^5/_2\, kT\,(n_1 v_1 + n_2 v_2 - nv) , \tag{2.2}$$

where k is the Boltzmann constant.

Using Eqs. (1.1) and (1.3) we can convert (2.2) to the form

$$S_D = \frac{5kT}{2m_1}\left(\frac{m_1}{m_2} - 1\right) D\,\frac{d\rho_1}{dr} . \tag{2.3}$$

Substituting the solution of Eq. (1.9) in (2.3), we obtain

$$S_D = \frac{5kT}{2m_1}\left(\frac{m_1}{m_2} - 1\right) D\left(\rho_a - \rho_\infty\right) \frac{a}{r^2} \ . \tag{2.4}$$

The energy flux transferred by heat conduction of a vapor-gas mixture with thermal conductivity $\varkappa$ is

$$S_\varkappa = -\varkappa\, dT / dr \ . \tag{2.5}$$

The equation for energy transfer in the vapor-gas mixture at constant pressure, with the terms due to the mean mass flux and internal friction neglected, has the form

$$\left(\frac{d}{dr} + \frac{2}{r}\right)(S_r + S_D + S_\varkappa) = 0 \ . \tag{2.6}$$

To solve this equation we need to know S_a – the total energy flux on the phase interface – which can be found from a solution of the internal problem.

The energy transfer equation for $r < a$ has the form

$$\left(\frac{d}{dr} + \frac{2}{r}\right)\varkappa' \frac{dT}{dr} + q = 0 \ , \tag{2.7}$$

where q is the intensity of the internal heat sources per unit volume of drop.

Since we assume that the radius of the drop significantly exceeds the mean free path of the gas molecules we can ignore the temperature discontinuity near the surface [1]. For a given drop surface temperature T_a, and if $T \neq \infty$ when $r = 0$, we can easily obtain a solution of Eq. (2.7)

$$T' - T_a' = (qa^2 / 6\varkappa')(1 - r^2 / a^2) \ , \tag{2.8}$$

from which it follows that the heat flux per unit area of drop surface due to internal heat sources is

$$-\varkappa' (dT' / dr)_a = {}^1/_3\, qa \ . \tag{2.9}$$

The energy flux transferred by radiation, diffusion, and heat conduction in the vapor-gas medium is given for $r = a$ by the Eq. (2.9) with the energy flux due to the phase transition subtracted,

$$S_a = {}^1/_3 qa - Lj \ , \tag{2.10}$$

where L is the specific heat of vaporization.

It follows from Eq. (2.6) and boundary condition (2.10) that

$$-\varkappa \frac{dT}{dr} = \frac{qa^2}{3r^2} - 4\varepsilon\sigma T_\infty^3 (T_a - T_\infty) \frac{a^2}{r^2} - \frac{\gamma a (\rho_a - \rho_\infty)}{r_2}$$

$$\gamma = D\left[\frac{\rho}{\rho_2} L - \frac{5kT_\infty}{2m_1}\left(\frac{m_1}{m_2} - 1\right)\right]. \tag{2.11}$$

Since the difference in temperatures on the drop surface and at infinity is assumed to be relatively small, then $\varkappa$ and D can be considered constant and, in addition, ρ can be replaced by ρ_a. The solution of Eq. (2.11), which describes the temperature distribution in the vapor-gas medium, has the form

$$\varkappa (T - T_\infty) = \frac{qa^3}{3r} - 4\varepsilon\sigma T_\infty^3 (T_a - T_\infty) \frac{a^2}{r^2} - \frac{\gamma (\rho_a - \rho_\infty) a}{r} \ . \tag{2.12}$$

Thus, the difference between the drop surface temperature and the temperature of the medium at infinity is

$$T_a - T_\infty = \frac{qa^2 - 3\gamma(\rho_a - \rho_\infty)}{3(\varkappa + 4a\varepsilon\sigma T_\infty^3)}. \tag{2.13}$$

3. Variation of Drop Radius with Time. It follows from (1.11) and (2.13) that

$$\rho_a - \rho_\infty = \frac{(4a\varepsilon\sigma T_\infty^3 + \varkappa)(\rho_{s\infty} - \rho_\infty) + \beta\gamma\rho_{s\infty} qa^2 / 3T_\infty}{\varkappa + 4a\varepsilon\sigma T_\infty^3 + \beta\gamma\rho_{s\infty} / T_\infty}. \tag{3.1}$$

Thus, we can calculate the vapor flux density on the drop surface and obtain an equation for the variation of drop radius with time,

$$a\dot{a} = -\frac{\rho D}{\rho_2 \rho'}\left[\rho_{s\infty} - \rho_\infty + \beta\rho_{s\infty}\frac{qa^2/3T_\infty - \gamma(\rho_{s\infty} - \rho_\infty)/T_\infty}{\varkappa + 4a\varepsilon\sigma T_\infty^3 + \beta\gamma\rho_{s\infty}/T_\infty}\right]. \tag{3.2}$$

The solution of this equation is very laborious and, hence, it is better to consider only special cases.

In the initial stage of evaporation of a sufficiently large drop and also in the case where the vapor density at infinity is the same as the saturated vapor density, the evaporation of the drop conforms to the following law:

$$\frac{\dot{a}}{a} = \frac{\rho_{s\infty}\rho\beta q D}{3\rho_2\rho' T_\infty(\varkappa + 4a\varepsilon\sigma T_\infty^3 + \beta\gamma\rho_{s\infty}/T_\infty)}. \tag{3.3}$$

Then

$$a/a_0 - A\ln(a/a_0) = 1 - t/\tau_0,$$
$$A = \frac{\varkappa + \beta\gamma\rho_{s\infty}/T_\infty}{4a_0\varepsilon\sigma T_\infty^3}, \quad \tau_0 = \frac{12\rho_2\rho' a_0\varepsilon\sigma T_\infty^4}{\rho\rho_{s\infty} q\beta D}. \tag{3.4}$$

As the drop evaporates, the effect of internal heat release and radiative transfer from the drop surface decreases. For sufficiently large times the evaporation will be given by the formula

$$a\dot{a} = -\frac{\rho D}{\rho_2\rho'}(\rho_{s\infty} - \rho_\infty)\left(1 - \frac{\beta\gamma\rho_{s\infty}/T_\infty}{\varkappa + \beta\gamma\rho_{s\infty}/T_\infty}\right) \tag{3.5}$$

The drop radius if $\rho_{s\infty}$ and ρ_∞ are constant varies with time in the following way:

$$(a/a_0)^2 = 1 - t/\tau_\infty$$
$$\tau_\infty = a_0^2\rho_2\rho'(1 + \beta\gamma\rho_{s\infty}/\varkappa T_\infty)/2\rho D(\rho_{s\infty} - \rho_\infty) \tag{3.6}$$

If the vapor density is negligibly small in comparison with the gas density and heat transfer by radiation and diffusion is small in comparison with the heat flux due to the phase transition, then (3.6) becomes the same as Mason's formula [3].

In the case where there are no internal heat sources and energy transfer is due solely to radiation in a transparent medium and heat conduction, Eq. (3.2) should become the same as the equation in [4]. In [4], however, the corresponding formula was given in an obviously distorted form.

If the drop radius is large in comparison with the radiation mean free path the approximation of radiant heat conduction is applicable in the whole space occupied by the vapor-gas medium. This case can be obtained from (3.2) by putting $\varepsilon = 0$, $\varkappa_0 = \varkappa_r + \varkappa_g$:

$$a\dot{a} = -\frac{\rho D}{\rho_2\rho'}\left[\rho_{s\infty} - \rho_\infty + \beta\rho_{s\infty}\frac{{}^1\!/_3 qa^2 - \gamma(\rho_{s\infty} - \rho_\infty)}{\varkappa_0 T_\infty + \beta\gamma\rho_{s\infty}}\right]. \tag{3.7}$$

Whence it follows that

$$(a / a_0)^2 = \exp(-t/\tau_1) - (\tau_1/\tau_2)[1 - \exp(-t/\tau_1)]$$
$$\tau_1 = 3\varkappa_0 \rho' \rho_2 T_\infty (1 + \beta\gamma\rho_{s\infty}/\varkappa_0) / 2\rho\rho_{s\infty}\beta q D$$
$$\tau_2 = a_0^2 \rho' \rho_2 (1 + \beta\gamma\rho_{s\infty}/\varkappa_0) / 2\rho(\rho_{s\infty} - \rho_\infty) D \,. \tag{3.8}$$

Evaporation and growth of the drop occur in a time of the order of $\tau_1\tau_2/(\tau_1 + \tau_2)$.

In the case of supersaturation ($\rho_\infty > \rho_{S\infty}$) the growth and evaporation of the drop cease, as (3.5) shows, when the limiting radius

$$a = [3\varkappa_0 T_\infty (\rho_\infty - \rho_{s\infty}) / \beta q]^{1/2} \tag{3.9}$$

is attained. The limiting radius can be found from the general formula (3.2)

$$a = (6\varepsilon\sigma T_\infty^4 / \beta\rho_{s\infty} q)[\rho_{s\infty} - \rho_\infty + \sqrt{(\rho_{s\infty} - \rho_\infty)^2 + (\rho_{s\infty} - \rho_\infty)\rho_{s\infty}\varkappa\beta q / 12\varepsilon^2\sigma^2 T^7} \tag{3.10}$$

which becomes Eq. (3.9) when $\varepsilon = 0$.

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